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A Problem of Ramanujam †

Solve

$$x^2=a+y \quad \ldots \quad \text{(i)}$$
$$y^2=a+z \quad \ldots \quad \text{(ii)}$$
$$z^2=a+u \quad \ldots \quad \text{(iii)}$$
$$u^2=a+x \quad \ldots \quad \text{(iv)}$$

1. Suppose x, y, z, u are the roots of a biquadratic

$$t^4+p_1 t^3+p_2 t^2+p_3 t+p_4=0$$

We denote $\sum x^n$ by S_n.

Now $S_1=-p_1$

$S_2=4a+S_1=4a-p_1$ from the given equations ... (a)

Also $S_2+p_1 S_1+2p_2=0$

$\therefore$ Substituting for S_1, S_2 we have $p_2=\dfrac{p_1^2+p_1-4a}{2}$ ... (b)

2. Subtract (iii) from (i) and (iv) from (ii)

$x^2-z^2=y-u$; $y^2-u^2=z-x$

$\therefore$ $(x^2-z^2)(y^2-u^2)=(y-u)(z-x)$

or $xy+zy+ux+uz=-1$

But $\sum xy=p_2$

So $xz+uy=p_2+1$ ... (c)

3. $x^3=ax+xy$

$y^3=ay+yz$

Adding $S_3=aS_1+xy+zy+ux+uz$

$=-ap_1-1$ ... (d)

Also $x^2z^2=a^2+au+ay+uy$

$y^2u^2=a^2+az+ax+zx$

Adding, $x^2z^2+y^2u^2=2a^2+aS_1+uy+zx$

$(xz+uy)^2=2a^2+aS_1+p_2+1+2p_4$ $\quad [xyzu=p_4]$

$(p_2+1)^2=2a^2-ap_1+p_2+1+2p_4$

$\therefore$ $2p_4=p_2^2+p_2-2a^2+ap_1$ ... (e)

4. Evidently

$$(x^2-y^2)(y^2-z^2)(z^2-u^2)(u^2-x^2)=(x-y)(y-z)(z-u)(u-x)$$

or $(x+y)(y+z)(z+u)(u+x)=1$

or $(x^2z^2+u^2y^2+2xyzu)+\sum x^2yz=1$

$(p_2+1)^2+p_1 p_3-4p_4=1$ $\quad \{\sum x^2yz=p_1 p_3-4p_4\}$

Substituting for p_4 from (e)

$p_1 p_3-p_2^2+4a^2-2ap_1=0$ ... (f)

† Vide *Collected Papers of Srinivasa Ramanujan*, 1927, p. 332 Q. 722 Also J. I. M. S. Series I Vol. VII p. 240.

5. We know

$$S_3 + p_1 S_2 + p_2 S_1 + 3p_3 = 0$$

Substituting for S_3, S_2, S_1, p_2

we get

$$6p_3 = 3p_1^2 + p_1^3 - 10\, ap_1 + 2 \qquad \ldots \quad (g)$$

Substitute for p_3 from (g) in (f)

we get

$$p_1^4 - (4a-3)\; p_1^2 - 4p_1 = 0 \qquad \ldots \quad (h)$$

$$p_1\; \{p_1^3 - (4a-3)\; p_1 - 4\} = 0$$

The cubic in p_1 can be solved by the usual methods. p_1 known, p_2, p_3, p_4 can easily be discovered and the corresponding biquadratic in t can be framed. The biquadratic may be solved by usual methods for x, y, zu; For the particular value $p_1 = 0$ the biquadratic is

$$t^4 - 2a\; t^2 + \tfrac{1}{3}\; t + a^2 - a = 0$$

6. By employing the same methods, we can solve the system of equations

$$x^2 = a + y$$

$$y^2 = a + z$$

$$z^2 = a + x$$

much more rapidly than Ramanujan did. His is a very laborious method.

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