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A Problem of Ramanujam †

Solve

$$x^2 = a + y \quad \ldots \quad \text{(i)}$$
$$y^2 = a + z \quad \ldots \quad \text{(ii)}$$
$$z^2 = a + u \quad \ldots \quad \text{(iii)}$$
$$u^2 = a + x \quad \ldots \quad \text{(iv)}$$

1. Suppose x, y, z, u are the roots of a biquadratic

$$t^4 + p_1 t^3 + p_2 t^2 + p_3 t + p_4 = 0$$

We denote $\sum x^n$ by S_n.

Now $S_1 = -p_1$

$S_2 = 4a + S_1 = 4a - p_1$ from the given equations ... (a)

Also $S_2 + p_1 S_1 + 2p_2 = 0$

$\therefore$ Substituting for S_1, S_2 we have $p_2 = \dfrac{p_1^2 + p_1 - 4a}{2}$ ... (b)

2. Subtract (iii) from (i) and (iv) from (ii)

$x^2 - z^2 = y - u$; $y^2 - u^2 = z - x$

$\therefore$ $(x^2 - z^2)(y^2 - u^2) = (y - u)(z - x)$

or $xy + zy + ux + uz = -1$

But $\sum xy = p_2$

So $xz + uy = p_2 + 1$ ... (c)

3. $x^3 = ax + xy$

$y^3 = ay + yz$

Adding $S_3 = aS_1 + xy + zy + ux + uz$

$= -ap_1 - 1$ ... (d)

Also $x^2 z^2 = a^2 + au + ay + uy$

$y^2 u^2 = a^2 + az + ax + zx$

Adding, $x^2 z^2 + y^2 u^2 = 2a^2 + aS_1 + uy + zx$

$(xz + uy)^2 = 2a^2 + aS_1 + p_2 + 1 + 2p_4$ $\quad [xyzu = p_4]$

$(p_2 + 1)^2 = 2a^2 - ap_1 + p_2 + 1 + 2p_4$

$\therefore$ $2p_4 = p_2^2 + p_2 - 2a^2 + ap_1$ ... (e)

4. Evidently

$(x^2 - y^2)(y^2 - z^2)(z^2 - u^2)(u^2 - x^2) = (x - y)(y - z)(z - u)(u - x)$

or $(x + y)(y + z)(z + u)(u + x) = 1$

or $(x^2 z^2 + u^2 y^2 + 2xyzu) + \sum x^2 yz = 1$

$(p_2 + 1)^2 + p_1 p_3 - 4p_4 = 1$ $\quad \{\sum x^2 yz = p_1 p_3 - 4p_4\}$

Substituting for p_4 from (e)

$p_1 p_3 - p_2^2 + 4a^2 - 2ap_1 = 0$ ... (f)

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† Vide *Collected Papers of Srinivasa Ramanujan*, 1927, p. 332 Q. 722 Also J. I. M. S. Series I Vol. VII p. 240.

5. We know

$$S_3 + p_1 S_2 + p_2 S_1 + 3p_3 = 0$$

Substituting for S_3, S_2, S_1, p_2

we get

$$6p_3 = 3p_1^2 + p_1^3 - 10\, ap_1 + 2 \quad \dots \quad (g)$$

Substitute for p_3 from (g) in (f)

we get

$$p_1^4 - (4a-3)\, p_1^2 - 4p_1 = 0 \quad \dots \quad (h)$$

$$p_1\, \{p_1^3 - (4a-3)\, p_1 - 4\} = 0$$

The cubic in p_1 can be solved by the usual methods. p_1 known, p_2, p_3, p_4 can easily be discovered and the corresponding biquadratic in t can be framed. The biquadratic may be solved by usual methods for x, y, zu; For the particular value $p_1 = 0$ the biquadratic is

$$t^4 - 2a\, t^2 + \tfrac{1}{3}\, t + a^2 - a = 0$$

6. By employing the same methods, we can solve the system of equations

$$x^2 = a + y$$

$$y^2 = a + z$$

$$z^2 = a + x$$

much more rapidly than Ramanujan did. His is a very laborious method.

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